CS1802 Week 8: Advanced Counting

Counting with Sets

i. Prove that \(|A| + |B| + |A \cap B \cap C| \geq |A \cap B| + |A \cap C| + |B \cap C|\)

ii. (difficulty ★) In a class of 20 students, each student has at least 14 friends (friends are reciprocal). Show that there are 4 students that form a clique, that is all 4 are pairwise friends.
Why is Jimmy’s solution wrong?

i. HW3 PB3 part A) In how many ways one can arrange symbols (a,b,c,d,e,f,g) into 5 bins preserving relative symbols order? Solution: Jimmy see this as balls in bins, sort of, but he uses partition/sum rule to break the problem into several disjoint cases, count them separately, and add up.

A case corresponds to exactly how many bins are non-empty; for each case, there are exactly 3(non-empty bins)-1 separators which can be anywhere in between the symbols, that is in 8 possible spots: |a|b|c|d|e|f|g|. Here are the cases:
- all symbols to 1 bin, thus 0 separators. Choosing the bin \( \binom{5}{1} \); choosing the separators \( \binom{8}{0} \)
- all symbols to 2 bins thus 1 separators. Choosing the bins \( \binom{5}{2} \); choosing the separators \( \binom{8}{1} \)
- all symbols to 3 bins so 2 separators. Choosing the bins \( \binom{5}{3} \); choosing the separators \( \binom{8}{2} \)
- all symbols to 4 bins so 3 separators. Choosing the bins \( \binom{5}{4} \); choosing the separators \( \binom{8}{3} \)
- all symbols to 5 bins so 4 separator. Choosing the bins \( \binom{5}{5} \); choosing the separators \( \binom{8}{4} \)

Applying product rule for each case then partition rule across cases gives \( \sum_{k=1}^{5} \binom{5}{k} \binom{8}{k-1} \).

This is incorrect, why?

How can it be fixed following initial idea to break into cases by number of non-empty bins?
ii. Count all passwords of exactly 8 capital letters that have a letter occurring at least 5 times. Examples: SAATARAA, TUUURUUE, ABABABB. Jimmy’s solution:
- choose a letter to repeat → 26 choices
- choose 5 places to put it → \( \binom{8}{5} \) choices
- fill the other 3 spaces with any 3 letters → 26^3 choices
Product rule gives 26*\( \binom{8}{5} \)* 26^3. Why is this incorrect? How can one count correctly?

iii. 10 men who are pairs of brothers \((a_1b_1, a_2b_2, a_3b_3, a_4b_4, a_5b_5)\) are to blind-date 10 women who are pairs of sisters \((x_1y_1, x_2y_2, x_3y_3, x_4y_4, x_5y_5)\) such that any two brothers do not date two corresponding sisters, that is for example if \((a_2, x_4)\) is a date then \(b_2\) cannot date \(y_4\).
In how many ways can the dates be arranged?
Jimmy’s solution: There are 10! ways to arrange the dates without restrictions. There are 5! * 2^5 ways to arrange dates that violates the restriction since it comes down permuting the pairs and then choosing for each pair which brother dates which sister (2 possibilites per pair). So the answer is 10! – 5! * 2^5. Why is this wrong?
iv. (difficulty ★) Virgil has a solution, also wrong: We need the number of derangements = permutations without fix point for n=5. Examples: 21453, 41253. Not a derangement: 52134 because 2 is in original position. For n=5 there are $D_5 = 44$ derangements which can be counted by brute force or by Inclusion-Exclusion (next exercise). Then the answer is 5! (choose a permutation of the 5 men $a_1..a_5$) * $2^5$ (choose which sister to date) * $D_5$ (choose a derangement for brothers $b_1..b_5$).

Why is Virgil’s solution wrong?
i. (difficulty ★) A derangement of 1 2 3 \ldots n is a permutation that leaves none of these numbers in place. By inspection, the derangements of 123 are 312 and 231. Find the number of derangements of 1 2 3 4 5 using Inclusion-Exclusion
ii. In an effort to address religious differences, 8 Christians ($c_1, \ldots, c_8$), and 8 Muslims ($m_1, m_2, \ldots, m_8$) sit at a round discussion table. In how many different ways can they sit such that every two adjacent people have different religions? In here “different ways” refers to at least one person having different left or right neighbors.

iii. (difficulty ★) Same problem with 8 Christians ($c_1, \ldots, c_8$), 6 Hindus ($h_1, \ldots, h_6$), and 3 Muslims ($m_1, m_2, m_3$).
iv. If we expand the trinomial \((x + y + z)^{10}\), what is the coefficient of term \(x^3 y^2 z^5\)?

v. How many integers from 00001 to 99,999 have no digit value occurring more than twice? Integers here are explicitly written on 5 digits with leading 0-s, so 00102 is not valid because 0 occurs 3 times; 62343 is a valid integer; 12117 is not because digit “1” appears 3 times.
EXTRA (difficulty ⭐ ⭐). A permutation of values 1..2n is is disturbance if two consecutive items have difference in values at least n. For example n=5 the sequence (5,8,9,3,1,4,2,6,10,7) is a disturbance due to consecutive items (9,3) whose difference is 6 ≥ n. Prove that out of all (2n)! permutations more than half are disturbances.

EXTRA (difficulty ⭐ ⭐). In how many ways one can pick a multiset of three positive numbers smaller than 100 with their sum a multiple of 20? Repetitions are possible, such as {18, 11, 11}, but order is irrelevant so {18, 11, 11} = {11, 18, 11}.
EXTRA (difficulty ★★★) What is the correct count for the brothers-sisters blind-date question?

EXTRA (difficulty ★) Consider an equilateral triangle of side length $n$, which is divided into unit triangles, as shown. A valid path runs from the triangle in the top row to the middle triangle in the bottom row, such that adjacent triangles in our path share a common edge and the path never travels up (from a lower row to a higher row) or revisits a triangle.

An example of one such path is illustrated below for $n = 5$. Compute the the number of paths. Hint: Construct a bijective mapping between valid paths and ordered lists of positive integers $(a_1, a_2, ..., a_n)$ with $a_i \leq i$. Then count the ordered lists.