Questions for chapter 2.1 and 2.2

1. (a) For the following switch diagrams, write the corresponding truth table and
decide whether they correspond to one of the logic gates AND or OR.

i.

ii.

(b) Draw switch diagrams for the following truth tables:

i. A | B | Output
   0  0  1
   0  1  1
   1  0  1
   1  1  0

ii. A | B | Output
     0  0  0
     0  1  1
     1  0  0
     1  1  0

(c) Logic circuit given; compute the output for each input given below:

i. I_0 = 0, I_1 = 0, I_2 = 0, I_3 = 0

ii. I_0 = 1, I_1 = 1, I_2 = 0, I_3 = 0

iii. I_0 = 1, I_1 = 1, I_2 = 1, I_3 = 1
2. In this question we will consider the parity of 3 bits. You may think of these 3 bits as if they represent an integer between 0 to 7.

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<th>Bit 1</th>
<th>Bit 2</th>
<th>Bit 3</th>
<th>Integer</th>
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We say that the parity of the bits is odd if exactly 1 or exactly 3 of the bits equal 1, and that the parity is even if exactly 0 or 2 of the bits equal 1. In the above table, for the integers 1, 2, 4, 7 the parity is odd, and for 0, 3, 5, 6 the parity is even. Throughout the question you may use AND gates, OR gates, and NOT gates.

(a) Given the value of 3 bits as input, design a logic circuit that determines whether the 3 bits are all 1. The output should be 1 if all the bits equal 1 and 0 otherwise. Your logic circuit should look like:

(b) Given the value of 3 bits as input, design a logic circuit that determines whether exactly 1 out of the 3 bits equals 1. The output should be 1 if exactly one bit equals 1 and 0 otherwise.

(c) Use the logic circuits that you designed in the previous sections to design a logic circuit that determines whether the parity of 3 bits is even or odd. The output should be 1 if the parity is odd and 0 otherwise.
I Logical Equivalence

1. \((A \land B) \lor (A \land \neg B)\) is logically equivalent to which of the following:

\[ A, B, 0, 1, A \lor (\neg B) \]

\textit{NOTE}: for the rest of the assignment, I will write AND, NOT, OR, etc rather than using the Boolean logic symbols.
Students: make sure to use the Boolean symbols in all assignments.

2. \(\neg(A \lor B)\) is logically equivalent to which of the following:

\[ A, B, 0, 1, (\neg A) \land (\neg B) \]

3. Use laws of logic to show that \(\neg(A \land (B \lor \neg A))\) and \(\neg A \lor \neg B\) are logically equivalent.

4. Evaluate \((A \oplus B) \land (B \lor (\neg A))\) (i) when A is T (true) and B is F (false); (ii) when A is F and B is T; and (iii) when A and B are both F.
5. How many total rows will there be in the truth table representation of 

\[(A \text{ XOR } B) \text{ OR } ((\text{NOT } C) \text{ AND } A)\]

6. What value of \( A \) will cause \((A \text{ XNOR } B) \text{ XOR } (A \text{ OR } B)\) to be \( T \) when \( B \) is \( T \)?

II  DNF and CNF

Given the truth table:

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1. Write an expression in DNF that has this truth table.

2. Write an expression in CNF for this truth table.
III For Fun

1. You are on Smullyan island where each inhabitant belongs to one of two clans: the TRUTH TELLERS who always tell the truth and the LIARS who always lie. You meet a couple. The husband says “Exactly one of us is lying.” The wife says “At least one of us is telling the truth.” Who, if anyone, is telling the truth?

Construct a truth table with two variables, H (meaning that the husband is a truth teller) and W (the wife is a truth teller) and two statement columns, one for the statement “Exactly one is lying”, the other for the statement “At least one is truthful.” Explain how your truth table can be used to discover who (if anyone) is telling the truth. (Note: you might also solve the problem by going through cases in your head. The truth table is used to organize this process.)

2. You are still on Smullyan island walking back to your hotel when the road forks into two branches. You don’t know which branch to take but there is an inhabitant sitting on a bench nearby and you are allowed to ask him a single question. What question could you ask to discover the way back? (Note: though all this happens on Smullyan island, this problem isn’t of the same kind as the one above).
CS1802 Week 3: Logic and Proofs

More Logic

i. Let P mean “it is raining,” Q mean “the sprinklers are on,” and R mean “the lawn is wet.” Phrase each English statement as propositional logic using $\lor$, $\land$, $\neg$, $\implies$, and $\iff$.

1. If it is raining, the lawn is wet.

2. If it is raining or the sprinklers are on, the lawn is wet.

3. If the lawn is not wet, it is not raining.

4. The lawn is wet if and only if it is either raining or the sprinklers are on.

5. “If the lawn is not wet, it is not raining” is actually just another way of saying “If it is raining, the lawn is wet.” The two statements are equivalent; they are both only false if it is raining and the lawn is not wet. Come up with an equivalent statement to “If it is raining or the sprinklers are on, the lawn is wet” that begins with “If the lawn is not wet . . .” and give the propositional logic form of this statement.

ii. Interpret the following statements as English sentences, then decide whether those statements are true if $x$ and $y$ can be any integers.

1. $\forall x \exists y : x + y = 0$

2. $\exists y \forall x : x + y = x$

3. $\exists x \forall y : x + y = x$
Proofs Interpret the following statements as English sentences, then decide whether those statements are true.

1. \( \forall x \in \mathbb{Z}, \exists y \in \mathbb{Z} : x + y = \text{prime} \)

2. \( \forall x \in \mathbb{Z}, \exists y \in \mathbb{Z} : x + y = \text{prime} \) and \( y = \text{prime} \)

3. \( \forall x \in \mathbb{Z}, x \text{ even}, \exists y \in \mathbb{Z} : x + y = \text{prime} \) and \( y = \text{prime} \)

4. \( \forall x \in \mathbb{R}, \forall y \in \mathbb{R} : x > y \iff x^2 > y^2 \)

5. \( \forall x \in \mathbb{R}, x > 0, \forall y \in \mathbb{R}, y > 0 : x > y \iff x^2 > y^2 \)

6. \( \forall x \in \mathbb{R}, x \neq 0, \forall y \in \mathbb{R} : x + y < 0 \iff x \cdot y < 0 \)
**Bitwise** Interpret the following statements as English sentences, then decide whether those statements are true. Variables $a, b, c$ are boolean and variables $A, B, C$ are boolean sequences (or sequences of bits), for example $A = a_1a_2...a_k = a_1, a_2, ..., a_k$ is a sequence of $k$ bits or $k$ boolean variables. $k$ is fixed (for example $k=32$).

$A, B, C$ allow boolean operations "bitwise": $A = a_1a_2...a_k$ and $B = b_1b_2...b_k$ then

$A \land B = a_1 \land b_1, a_2 \land b_2, ..., a_k \land b_k$

Similarly $A \lor B = a_1 \lor b_1, a_2 \lor b_2, ..., a_k \lor b_k$, etc

1. $\forall a, b, c \in \{T, F\} : a \land b = a \land c \Rightarrow b = c$

2. $\forall A, B, C \in \{T, F\}^k : A \land B = A \land C \Rightarrow B = C$

3. $\exists A, \forall B, C \in \{T, F\}^k : A \land B = A \land C \Rightarrow B = C$

4. $\forall a, b, c \in \{T, F\} : a \lor b = a \lor c \Rightarrow b = c$

5. $\forall A, B, C \in \{T, F\}^k : A \lor B = A \lor C \Rightarrow B = C$

6. $\exists A, \forall B, C \in \{T, F\}^k : A \lor B = A \lor C \Rightarrow B = C$

7. $\forall a, b, c \in \{T, F\} : a \text{ XOR } b = a \text{ XOR } c \Rightarrow b = c$

8. $\forall A, B, C \in \{T, F\}^k : A \text{ XOR } B = A \text{ XOR } C \Rightarrow B = C$
9. This last XOR equation should allow you to prove the strategy for the XOR-square-game: the player wants to leave the table such that when the counts are written in binary, their XOR across all rows is 0.

**Prove** that if player A leaves a board with XOR(binary rows)=0, then player B cannot do the same.

**Prove** that if player B leaves a board with XOR(binary rows)≠0, then player A has a move to make the board as desired XOR(binary rows)= 0.
10 wise men

10 wise men leave in a village; each man has a color dot on the forehead either R or B not known to him; knowing his color means immediate death. But everyone knows the other men’s colors, so B person sees 5R and 4B.

The men don’t speak/communicate to each other, but each morning they meet in a circle and they can see if anyone died. They are extremely smart (can infer anything) and know when someone dies it’s because he must have figured out his color.

For quite a few days this goes unchanged, until one day a stranger passes to the village and remarks ”the number of B colors is not 20”. Prove that eventually everyone in the village will figure out his color and die.