CS1802 Weeks 12-13

Recurrences

1. If mergesort divided its input array into five pieces instead of two, calling mergesort on each piece and combining with a linear-time 5-way merge, what would its recurrence be?

Solution: The recurrence would be $T(N) = 5T(N/5) + cN$. The recursive calls are a fifth the size and there are five of them, but the merge is still linear time.

2. Find $T(n)$ for $n = 16$ if $T(1) = 1$ and $T(N) = 4T(N/2) + 1$.

Solution: $T(1) = 1$
$T(2) = 4T(1) + 1 = 4 + 1 = 5$
$T(4) = 4T(2) + 1 = 4*5 + 1 = 21$
$T(8) = 4T(4) + 1 = 4*21 + 1 = 85$
$T(16) = 4T(8) + 1 = 4*85 + 1 = 341$
3. Solve the previous \( T(N) \) recurrence for its order of growth. (You can approximate and ignore constants).

Solution: The way we tend to solve such problems in this course is through repeated substitution of the recursive formula into itself, to try to guess a pattern. Here this repeated substitution gives the pattern:

\[
4T(N/2) + 1 \\
= 4[4T(N/4) + 1] + 1 \\
= 16T(N/4) + 4 + 1 \\
= 16[4T(N/8) + 1] + 4 + 1 \\
= 64T(N/8) + 16 + 4 + 1 \\
= \ldots
\]

From this pattern, we can guess the formula \( 4^i T(N/2^i) + \sum_{j=0}^{i} 4^j \), which bottoms out when \( i = \log_2 N \). Substituting gives \( \sum_{i=0}^{\log_2 N} 4^i + 4^{\log_2 N} \), and if we just concentrate on the largest term of the sum and ignore constants, that’s roughly \( 4^{\log_2 N} \) multiplied by a constant. That’s equal to \( (2^{\log_2 N})^2 = N^2 \).

4. EXTRA ★ Determine the order of growth of the recurrence

\[
T(N) = T(N - 1) + T(N - 2) + T(N - 3)
\]

SOLUTION (idea only): Similar to Fibonacci, we speculate that this recursion satisfies \( T(N) = \Theta(\beta^n) \) where \( \beta^3 = \beta^2 + \beta + 1 \)
Growth of Functions

1. Organize the following functions into six columns. Items in the same column should have the same asymptotic growth rates (they are big-O and big-Θ of each other). If a column is to the left of another column, all its growth rates should be slower than those of the column to its right.

\[ n^2, \, n!, \, n \log_2 n, \, 3n, \, 5n^2 + 3, \, 2^n, \, 10000, \, n \log_3 n, \, 100, \, 100n \]

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<table>
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<tbody>
<tr>
<td>100</td>
<td>3n</td>
<td>( n \log_2 n )</td>
<td>( n^2 )</td>
<td>( 2^n )</td>
</tr>
<tr>
<td>10000</td>
<td>100n</td>
<td>( n \log_3 n )</td>
<td>( 5n^2 + 3 )</td>
<td>( n! )</td>
</tr>
</tbody>
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Solution:

2. Using the definition of big-O, show \( 100n + 5 = O(2n) \).

Solution: Let \( c = 1000 \) (somewhat arbitrarily) and \( n_0 = 1 \) (also a little arbitrary). It’s clear that \( 100n + 5 \leq 2000n \) for all \( n \geq 1 \), so this shows \( 100n + 5 \) is big-O of \( 2n \).
3. Using the definition of big-O, is it true that \( n = O(2^n) \)?

**Solution:** Yes, technically, since \( c = 1 \) and \( n_0 = 0 \) have \( n \leq 1 \cdot 2^n \) for all \( n \geq 0 \). Big O is like a “less than or equal” relation, not equality.

4. True or false and explain: If \( f(n) = \Omega(g(n)) \), then \( g(n) = O(f(n)) \).

**Solution:** True. If \( g(n) \) is an asymptotic lower bound for \( f(n) \), then \( f(n) \) is an asymptotic upper bound for \( g(n) \).

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**Graphs**

1. In a complete graph (a graph with all possible edges), how many possible cycles are there that visit all \( V \) vertices and return to the start? (Assume the same basic route with a different start vertex is a different cycle.)
2. In a graph with $V$ vertices where every vertex has degree 4, how many edges are in the graph?

   **Solution:** $2V$ since each edge contributes 2 to the total number of vertices.

3. The complement $G'$ of a graph $G$ is the graph where if two vertices share an edge in $G$, there is no edge between them in $G'$; and if they didn’t have an edge in $G'$, they do now. Figure out a formula for the number of edges in the complement of a graph with $V$ vertices and $E$ edges.

   **Solution:** The complement contains a number of edges equal to the number in the complete graph minus the number of original edges, or ${V \choose 2} - E$. 

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*Solutions to previous problems:*

- **Solution:** $n!$
4. If negative weights are possible, can the minimal weight subgraph that connects all the vertices in a graph contain a cycle? What if negative and zero weights are not possible?

Solution: Yes, one could imagine a ring graph in which every weight is negative, for example; adding any edge decreases the weight, so the minimal weight subgraph is itself a cycle. If negative and zero weight edges are not possible, then cycles in the minimal weight subgraph are not possible, because removing an edge from the cycle would leave the vertices connected with a smaller weight. Thus in this case, minimal spanning subgraphs are always minimal spanning trees.
Give asymptotic upper and lower bounds for $T(n)$ in each of the following recurrences. Assume that $T(n)$ is constant for $n \leq 2$. Make your bounds as tight as possible, and justify your answers.

a. $T(n) = 2T(n/2) + n^4$.

b. $T(n) = T(7n/10) + n$.

c. $T(n) = 16T(n/4) + n^2$.

d. $T(n) = 7T(n/3) + n^2$.

e. $T(n) = 7T(n/2) + n^2$.

f. $T(n) = 2T(n/4) + \sqrt{n}$.

g. $T(n) = T(n-2) + n^2$. 
Give asymptotic upper and lower bounds for $T(n)$ in each of the following recurrences. Assume that $T(n)$ is constant for sufficiently small $n$. Make your bounds as tight as possible, and justify your answers.

a. $T(n) = 4T(n/3) + n \lg n$.

b. $T(n) = 3T(n/3) + n/\lg n$.

c. $T(n) = 4T(n/2) + n^2 \sqrt{n}$.

d. $T(n) = 3T(n/3 - 2) + n/2$.

e. $T(n) = 2T(n/2) + n/\lg n$.

f. $T(n) = T(n/2) + T(n/4) + T(n/8) + n$.

g. $T(n) = T(n - 1) + 1/n$.

h. $T(n) = T(n - 1) + \lg n$.

i. $T(n) = T(n - 2) + 1/\lg n$.

j. $T(n) = \sqrt{n}T(\sqrt{n}) + n$. 
EXTRA Graphs

1. ★ Prove that if $G$ is a graph with at least 6 vertices, then $G$ or $G$-complement contains a cycle of length 3. HINT: consider all possible edges rom a fixed vertex. How many of them are guaranteed to be in one of the graphs?

Solution:

SOLUTION A:
There are $\binom{6}{2} = 15$ total edges among $G$ and $G'$ so (Pigeonhole P) one of them, say $G$, has at least 8 edges. But a graph with 6 nodes and 8 edges either has a 3-cycle (DONE) or if not, then it is bipartite 4-vs-2 or 3-vs-3; in either of these cases $G'$ has complete set of edges in a partition of 3 or 4, which means a 3-cycle.

SOLUTION B:
Consider all edges from vertex A to the other vertices. They are 5, so by PP at least 3 of them have to be in either G or G-complement. Say three edges AB AC AD are in $G$.
Then either one of the BCD edges is in G, so G contains a 3-cycle, or all three edges BCD are in G-complement thus a 3-cycle.

2. Prove that if $G$ is a graph with all vertices of degree $d$ or higher, then $G$ contains a path of length $d$.

Solution: Color all edges white (available). Start in some vertex $v_0$ and color red the edges used (walked on). Since $\deg(v_0) \geq d$ walk on one of the white edges to a vertex $v_1$, color that edge red. Since we only colored one edge so far and $\deg(v_1) \geq d$ there must be at least one other edge incident on $v_1$ that is white, walk on it to $v_2$ etc.
In general: after walking to $v_k (k < d)$ we colored $k$ edges red. Since $k < d$ and $\deg(v_k) \geq d$ then $v_k$ must have an incident edge still white, so we walk on this edge to $v_{k+1}$ and color it red. This works until we hit $v_d$ so we have a path of length $d$. 

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3. You are given $V$ integers $d_1, d_2, ..., d_V > 0$ that satisfy $\sum_{v \in V} d_v = 2(|V| - 1)$. Explain how to construct a tree with $V$ vertices that have these exact degrees. Hint: by induction over $V$.

**Solution:** Construction goes by induction. Base case: it works for $V=1$, $V=2$
Induction Step from $V$ to $V+1$: Suppose it works for $V$, that is if $V$ integers satisfy the property then there is a tree with them as degrees. Then we show this true for $V+1$
Proof:
Consider integers $d_1, d_2, ..., d_{V+1} > 0$ that satisfy $\sum_{v \in V} d_v = 2(|V + 1| - 1)$
First: find one of the integers $d_{V+1} = 1$ and another $d_V > 1$ (I resorted the indices so these two are the last two). Now consider the following $V$-1 integers: $d_1, d_2, ..., d_{V-1}, d_V - 1$ which are the original ones with two changes $d_V$ is down by 1, and $d_{V+1}$ is missing. These $V$ integers satisfy the condition for $V \sum_{v \in V} d_v = 2(|V| - 1)$ so by induction hypothesis there is a tree $T$ with these degrees.
To that tree we add an edge from $V$ to leaf $V + 1$ which fixes these two degrees to the original values. We have now a tree with degrees as required for the input $V + 1$ integers.

4. ★★ In the group of 10 people, everyone knows at least 5 other people. Prove that they can seat at a round table in such a way that everyone knows the two people sitting next to them. Hint: The problem is asking for a Hamiltonian cycle, lookup “Dirac’s” theorem and make the argument for $n = 10$.

**Solution:** [https://en.wikipedia.org/wiki/Hamiltonian_path#BondyChvatal_theorem](https://en.wikipedia.org/wiki/Hamiltonian_path#BondyChvatal_theorem)