Recurrences

1. If mergesort divided its input array into five pieces instead of two, calling mergesort on each piece and combining with a linear-time 5-way merge, what would its recurrence be?

2. Find $T(n)$ for $n = 16$ if $T(1) = 1$ and $T(N) = 4T(N/2) + 1$. 
3. Solve the previous $T(N)$ recurrence for its order of growth. (You can approximate and ignore constants).

4. EXTRA ★ Determine the order of growth of the recurrence $T(N) = T(N - 1) + T(N - 2) + T(N - 3)$
Growth of Functions

1. Organize the following functions into six columns. Items in the same column should have the same asymptotic growth rates (they are big-O and big-Θ of each other). If a column is to the left of another column, all its growth rates should be slower than those of the column to its right.

\( n^2, \ n!, \ n \log_2 n, \ 3n, \ 5n^2 + 3, \ 2^n, \ 10000, \ n \log_3 n, \ 100, \ 100n \)

2. Using the definition of big-O, show \( 100n + 5 = O(2n) \).

3. Using the definition of big-O, is it true that \( n = O(2^n) \)?
4. True or false and explain: If \( f(n) = \Omega(g(n)) \), then \( g(n) = O(f(n)) \).

Graphs

1. In a complete graph (a graph with all possible edges), how many possible cycles are there that visit all \( V \) vertices and return to the start? (Assume the same basic route with a different start vertex is a different cycle.)

2. In a graph with \( V \) vertices where every vertex has degree 4, how many edges are in the graph?
3. The complement $G'$ of a graph $G$ is the graph where if two vertices share an edge in $G$, there is no edge between them in $G'$; and if they didn’t have an edge in $G'$, they do now. Figure out a formula for the number of edges in the complement of a graph with $V$ vertices and $E$ edges.

4. If negative weights are possible, can the minimal weight subgraph that connects all the vertices in a graph contain a cycle? What if negative and zero weights are not possible?
EXTRA Recurrences 1

Give asymptotic upper and lower bounds for $T(n)$ in each of the following recurrences. Assume that $T(n)$ is constant for $n \leq 2$. Make your bounds as tight as possible, and justify your answers.

a. $T(n) = 2T(n/2) + n^4$.

b. $T(n) = T(7n/10) + n$.

c. $T(n) = 16T(n/4) + n^2$.

d. $T(n) = 7T(n/3) + n^2$.

e. $T(n) = 7T(n/2) + n^2$.

f. $T(n) = 2T(n/4) + \sqrt{n}$.

g. $T(n) = T(n - 2) + n^2$.
Give asymptotic upper and lower bounds for $T(n)$ in each of the following recurrences. Assume that $T(n)$ is constant for sufficiently small $n$. Make your bounds as tight as possible, and justify your answers.

a. $T(n) = 4T(n/3) + n \lg n$.

b. $T(n) = 3T(n/3) + n/\lg n$.

c. $T(n) = 4T(n/2) + n^2 \sqrt{n}$.

d. $T(n) = 3T(n/3 - 2) + n/2$.

e. $T(n) = 2T(n/2) + n/\lg n$.

f. $T(n) = T(n/2) + T(n/4) + T(n/8) + n$.

g. $T(n) = T(n - 1) + 1/n$.

h. $T(n) = T(n - 1) + \lg n$.

i. $T(n) = T(n - 2) + 1/\lg n$.

j. $T(n) = \sqrt{n}T(\sqrt{n}) + n$. 
EXTRA Graphs

1. ★ Prove that if $G$ is a graph with at least 6 vertices, then $G$ or $G$-complement contains a cycle of length 3.

2. Prove that if $G$ is a graph with all vertices of degree $d$ or higher, then $G$ contains a path of length $d$.

3. You are given $V$ integers $d_1, d_2, ..., d_V > 0$ that satisfy $\sum_{v \in V} d_v = 2(|V| - 1)$. Explain how to construct a tree with $V$ vertices that have these exact degrees. Hint: by induction over $V$. 
4. ★★ In the group of 10 people, everyone knows at least 5 other people. Prove that they can seat at a round table in such a way that everyone knows the two people sitting next to them. Hint: The problem is asking for a Hamiltonian cycle, lookup “Dirac’s” theorem and make the argument for $n = 10$. 
