i. If \( A = \{1, 2, 3\} \) and \( B = \{a, b\} \), how many elements are in \( \mathcal{P}(A \times B) \)?

\[
|A \times B| = |A| \cdot |B| = 3 \cdot 2 = 6
\]

\[
|\mathcal{P}(A \times B)| = 2^{|A \times B|} = 2^6 = 64
\]

ii. If $200,001$ is distributed among all 50 states of the U.S. (fractions of a dollar aren’t allowed), and we analyze the state that received the largest amount of money, what is the minimum number of dollars this state must have received? Explain.

\[
\left\lceil \frac{200001}{50} \right\rceil = \left\lceil 4000.02 \right\rceil = 4001
\]

(by pigeonhole principle

(assuming no cents, only whole dollars)

1010

iii. How many ways are there to rearrange the sequence of letters in the word FACEBOOK? (You don’t need to simplify.)

\[
\frac{8!}{2!} = 3360
\]

for the 2 "O"s
removes duplicate arrangements
since "O"s are identical
iv. How many numbers are there between 1 and 99999 inclusive whose digits sum to 8? Compute your answer as a decimal integer.

\[ \binom{8 + 5 - 1}{5 - 1} = \binom{12}{4} = \frac{12!}{4!8!} = \frac{12 \cdot 11 \cdot 10 \cdot 9}{4 \cdot 3 \cdot 2} = 495 \]

v. Rolling four 6-sided dice, what is the probability of having three dice show one number and one die show another? (You don't need to simplify or multiply out your answer.)

Choose one number
Choose a number which are same
Choose second number

\[ \binom{4}{3} \cdot \binom{6}{1} \cdot \binom{1}{1} \]

\[ \binom{6}{4} \]

\[ \text{all possible outcomes} \]

vi. In a 52-card deck (fours suits times thirteen values), what is the expected number of aces in a five-card hand? You don't need to simplify.

\[ X = \text{# of aces} \]

\[ E(X) = 1 \cdot \frac{4}{52} \cdot (\frac{48}{51}) + 2 \cdot \frac{4}{52} \cdot (\frac{48}{51}) + 3 \cdot \frac{4}{52} \cdot (\frac{48}{51}) \]

\[ \frac{(52-4=48)}{(52)} \]

\[ \text{all possible hands} \]

\[ \binom{4}{1} \cdot \binom{4}{1} \cdot \binom{4}{1} \cdot \binom{4}{1} \cdot \binom{4}{1} \]

\[ \binom{52}{5} \]
i. If $A = \{1, 2, 3, 4\}$ and $B = \{d, e, f\}$, how many elements are in $\mathcal{P}(A \times B)$?

\[
\mathcal{P}(\{1, 2, 3, 4\} \times \{d, e, f\}) = \{\{1, d\}, \{1, e\}, \{1, f\}, \{2, d\}, \{2, e\}, \{2, f\}, \{3, d\}, \{3, e\}, \{3, f\}, \{4, d\}, \{4, e\}, \{4, f\}\}
\]

12 items in the set, so $2^{12}$

ii. If $150,001$ is distributed among all 50 states of the U.S. (fractions of a dollar aren't allowed), and we analyze the state that received the largest amount of money, what is the minimum number of dollars this state must have received? Explain.

\[
\frac{3,000}{150,000} = 0.02
\]

\[
\frac{3,001}{150,000} = 0.02
\]

$3,000$ because all other states get $3,000$ and there $1$ left over for the remaining state. Pigeon hole principle.

iii. How many ways are there to rearrange the sequence of letters in the word FACEBOOK? (You don't need to simplify.)

\[
\frac{8!}{2} = 10,080
\]

Divide by 2 because there are two Os.
iv. How many numbers are there between 1 and 99999 inclusive whose digits sum to 6? Compute your answer as a decimal integer.

\[ \binom{5+6-1}{6-1} = \binom{10}{4} \quad \frac{10!}{4!6!} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 210 \]

v. Rolling five 6-sided dice, what is the probability of having four dice show one number and one die show another? (You don't need to simplify or multiply out your answer.)

\[ \binom{5}{1} \left( \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{5}{6} \right) \]

\[ \frac{5 \cdot 6 \cdot 5}{6^5} = \frac{5^2}{6^4} \]

vi. In a 52-card deck (four suits times thirteen values), what is the expected number of aces in a five-card hand? You don't need to simplify.

\[ \frac{4}{52} = \frac{x}{5} \]

\[ 52x = 20 \]

\[ x = \frac{20}{52} = \frac{10}{26} = \frac{5}{13} \]

\[ \frac{20}{52} \text{ of new aces is expected} \]
i. If $A = \{a, b, c\}$ and $B = \{d, e, f\}$, how many elements are in $\mathcal{P}(A \times B)$?

\[ \mathcal{P}(A \times B) = 2^{1 \times 3} = 2^9 = 512 \]

\[ A \times B = \{ (a, d), (a, e), (a, f), (b, d), (b, e), (b, f), (c, d), (c, e), (c, f) \} \]

\[ |A \times B| = 9 \]

ii. If $100,001$ is distributed among all 50 states of the U.S. (fractions of a dollar aren't allowed), and we analyze the state that received the largest amount of money, what is the minimum number of dollars this state must have received? Explain.

Pigeon hole principle with 100,001 pigeons and 50 holes.

\[ \frac{100,001}{50} \geq 2,001 \]

$2,001$ is the minimum.

iii. How many ways are there to rearrange the sequence of letters in the word EXPECTATION? (You don't need to simplify.)

\[ \frac{11!}{2 \times 2} \]

$E$ is counted twice.

$T$ is counted twice.
iv. How many numbers are there between 1 and 9999 inclusive whose digits sum to 9? Compute your answer as a decimal integer.

\[
\begin{align*}
9 \text{ balls in 4 bins} & \quad (n + (k-1)) = \binom{12}{3} = \frac{12!}{9! \cdot 3!} = \frac{12 \cdot 11 \cdot 10}{2 \cdot 2} = \frac{2 \cdot 11 \cdot 10}{2} = 220
\end{align*}
\]

v. Rolling four 6-sided dice, what is the probability of having two dice show one number and the other two dice show another? (You don't need to simplify or multiply out your answer.)

\[
\binom{6}{2} - \text{two numbers the dice will show}
\]

\[
\begin{align*}
1122 & \quad \text{just like a roll of 4, 6 and 6, 4 amount} \\
1212 & \quad \text{the same, these rolls amount the same} \\
1221 & \\
2211 & \\
2121 & \\
2112 & \quad \text{6 ways}
\end{align*}
\]

vi. In a 54-card deck (52 cards and two jokers), what is the expected number of jokers in a five-card hand? You don't need to simplify.

\[
\begin{align*}
\frac{1}{\binom{54}{5}} \cdot \binom{52}{2} \cdot \binom{3}{1} \cdot \binom{5}{2} \cdot 0.2 \\
\frac{1}{\binom{54}{5}} \cdot \binom{52}{2} \cdot \binom{3}{1} \cdot \binom{5}{3} \cdot 0.1
\end{align*}
\]
i. If $A = \{a, b, c\}$ and $B = \{d, e, f\}$, how many elements are in $\mathcal{P}(A \times B)$?

\[
|A| = 3 \quad |B| = 3 \quad |A \times B| = 3 \cdot 3 = 9
\]

\[
|\mathcal{P}(A \times B)| = 2^{|A \times B|} = 2^9 = 512
\]

ii. If $50,001$ is distributed among all 50 states of the U.S. (fractions of a dollar aren’t allowed), and we analyze the state that received the largest amount of money, what is the minimum number of dollars this state must have received? Explain.

If we want the minimum largest amount, we must distribute $50,001$ as evenly as possible. If we do this, each state gets 1000 dollars, but there is 1 leftover dollar that must be given to a state. So, 1 state must get at least 1001 dollars. This happens because of the pigeon hole principle.

iii. How many ways are there to rearrange the sequence of letters in the word EXPECTATION? (You don’t need to simplify.)

\[
\begin{align*}
2 & \quad E \quad Z \quad T \quad 1 \quad N \\
1 & \quad X \quad 1 \quad A \\
1 & \quad P \quad 1 \quad I \\
1 & \quad C \quad 1 \quad O
\end{align*}
\]

\[
\binom{11}{2}(9)(2) \cdot 7!
\]

↑ choose 2 spots for E
↑ ways to arrange remaining letters
↑ choose 2 spots for T

\[
\binom{11}{2}(9)(2) \cdot 7!
\]
iv. How many numbers are there between 1 and 9999 inclusive whose digits sum to 6? Compute your answer as a decimal integer.

\[
\text{Distribute } 6 \text{ over 4 digits. For example } 0, 0, 0, 6 \text{ or } 0, 5, 1, 0. \text{ This is balls in bins, } 4 \text{ bins } \rightarrow 3 \text{ separators.}
\]

\[
\binom{6+4-1}{4-1} = \binom{9}{3} = \frac{9!}{6!3!} = \frac{9 \cdot 8 \cdot 7}{3 \cdot 2} = 84.
\]

v. Rolling five 6-sided dice, what is the probability of having three dice show one number and two dice show another? (You don’t need to simplify or multiply out your answer.)

\[
\binom{5}{3} \cdot \frac{1}{6} \cdot \binom{1}{1} \cdot \binom{1}{1} \cdot \binom{1}{1} \cdot \binom{5}{2} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6}.
\]

vi. In a 54-card deck (52 cards and two jokers), what is the expected number of jokers in a five-card hand? You don’t need to simplify.

\[
E[\text{Joker}] = \frac{2}{54}
\]

\[
E[\text{5 Jokers}] = E[\text{Joker}] + E[\text{Joker}] + E[\text{Joker}] + E[\text{Joker}] + E[\text{Joker}]
\]

\[
= 5 \cdot E[\text{Joker}]
\]

\[
= 5 \cdot \frac{2}{54} = \frac{10}{54}
\]