Last time
Induction
- weak
- started strong

Today
- Finish
  Strong induction
- Recurrences

Next time
- Finish Recurrences
Strong Induction

- Base case(s): e.g. $n=1$ (but often need multiple base cases)

- Ind. step: Show that if true if $k < n$
  then must be true at $n$.

\[
S(1) \Rightarrow S(2) \\
S(1), S(2) \Rightarrow S(3) \\
S(1), S(2), S(3) \Rightarrow S(4) \\
\vdots
\]
Postage stamp problem - claim: Postage for any \( n \geq 12 \) can be made with only 4 & 5¢ stamps.

Proof:

Ind. step: Show that if true for all \( 12 \leq k < n \), then true for \( n \).

Use 4¢ stamp. Then must make postage for \( n-4 \)¢.

Since \( n-4 < n \), this can be done w/ 4¢ & 5¢ stamps by ind. hyp. ✓

\[
\begin{array}{c|c|c}
\text{n} & \text{4¢} & \text{5¢} \\
\hline
12 & 3 & \\
13 & 2 & 1 \\
14 & 1 & 2 \\
15 & 3 & \\
\end{array}
\]

Base case:  
\( n = 12 \)  \( 3 \times 4¢ \)
\( E_2 = \) byte that does not contain consecutive 1s.

- generate 2 bits at a time \( \{00, 01\} \) allowed
- left-to-right
  - if 01, then the next two bits cannot be 10

<table>
<thead>
<tr>
<th>first 2 bits</th>
<th>next 2 bits</th>
<th>next 2 bits</th>
<th>last 2 bits</th>
</tr>
</thead>
<tbody>
<tr>
<td>00, 10</td>
<td>00, 10</td>
<td>00, 10</td>
<td>00, 01, 10</td>
</tr>
<tr>
<td>01</td>
<td>01</td>
<td>00</td>
<td>00, 01</td>
</tr>
<tr>
<td>01</td>
<td>00</td>
<td>00, 10</td>
<td>00, 01, 10</td>
</tr>
<tr>
<td>01</td>
<td>01</td>
<td>00</td>
<td>00, 01</td>
</tr>
<tr>
<td>01</td>
<td>00</td>
<td>00, 10</td>
<td>00, 01, 10</td>
</tr>
<tr>
<td>01</td>
<td>01</td>
<td>00</td>
<td>00, 01</td>
</tr>
<tr>
<td>01</td>
<td>00</td>
<td>00</td>
<td>00, 01, 10</td>
</tr>
<tr>
<td>01</td>
<td>00</td>
<td>01, 10</td>
<td>00, 01</td>
</tr>
</tbody>
</table>

\[ P(E_2) = \frac{55}{256} \approx 21.48\% \]
Q: How many bit strings of length $n$ have no consecutive 1s?

<table>
<thead>
<tr>
<th>$n$</th>
<th>Strings</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0, 1</td>
</tr>
<tr>
<td>2</td>
<td>00, 01, 10, 11</td>
</tr>
<tr>
<td>3</td>
<td>000, 001, 010, 011, 100, 101, 110, 111</td>
</tr>
<tr>
<td>4</td>
<td>0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111</td>
</tr>
</tbody>
</table>

Conjecture: The number of such strings $S(n) = F_{n+2}$ for $n \geq 1$.

Proof (Pf):

Inductive step: Show if true for $k < n$, then true at $n$.

- $S(k) = F_{k+2}$ for $k < n$ implies $S(n) = F_{n+2}$.

Every bit string of length $n$ with no consecutive 1s either begins with a 0 or a 1.

- If begins with 0:
  - How many? $S(n-1) = F_{n+1}$
  - Total = $F_{n+1} + F_n = F_{n+2}$

- If begins with 1:
  - Next bit must be 0.
  - How many? $S(n-2) = F_{n}$.
Since ind. hyp. relies on two cases back, i.e., to prove $S(n)$ we needed $S(n-1) \& S(n-2)$, need two base cases

$n=1 \checkmark$ \ } \text{ proven earlier.}

$n=2 \checkmark$
Merge Sort

\[ 3 + 25 \]
\[ 2 \ 3 \ 5 \ 7 \]
\[ 1 \ 4 \ 6 \ 8 \]
\[ \overbrace{12345678} \]

\[ T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{2}\right) + 15n + 6 \]

Reality:

\[ T(n) = 2T\left(\frac{n}{2}\right) + n \]

but asymptotically and in terms of order notations,

Just \[ T(n) = 2T\left(\frac{n}{2}\right) + n \]

\[ T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2T\left(\frac{n}{2}\right) + n & \text{if } n \geq 2 \end{cases} \]
\[ T(n) = 2T\left(\frac{n}{2}\right) + n \]

\[ T(\circ) = \circ + 2T(\frac{\circ}{2}) \]

\[ T(n) = n + 2T\left(\frac{n}{2}\right) \]
\[ = n + 2\left( \frac{n}{2} + 2T\left(\frac{n}{2^2}\right) \right) \]
\[ = 2n + 2^2 T\left(\frac{n}{2^2}\right) \]
\[ \vdots \]
\[ = 3n + 2^3 T\left(\frac{n}{2^3}\right) \]
\[ \vdots \]
\[ = k \cdot n + 2^k \cdot T\left(\frac{n}{2^k}\right) \]