Last time
- Finish sets & set operations
- Basic rules for counting
  - product rule
  - sum rule
  - principle of inclusion-exclusion

Today
- Pigeon hole principle
- Permutations & Combinations

Next time
- Balls-in-bins
- Binomial Theorem

Announcement
- Lecture Videos
- Lecture Notes
Pigeonhole Principle

If you place \( n+1 \) (or more) objects in \( n \) boxes, then at least one box has 2 or more objects.

Claim: Every integer \( n \) has a multiple that contains only 1's & 0's in decimal.

\[
\begin{align*}
\text{e.g.} & \quad 2 \to 10 \\
& \quad 4 \to 100 \\
& \quad 5 \to 10 \\
& \quad 3 \to ?
\end{align*}
\]

\[
\begin{align*}
\text{e.g.} & \quad 3 \\
& \quad 1 \mod 3 = 1 \\
& \quad 11 \mod 3 = 2 \\
& \quad 111 \mod 3 = 0 \\
& \quad 1111 \mod 3 = 1 \\
1111 = 370 \cdot 3 + 1 \\
-1 = 0 \cdot 3 + 1 \\
1110 = 370 \cdot 3
\end{align*}
\]

Proof: Consider the following #'s:

\[
1, 11, 111, 1111, 11111, \ldots, \underbrace{111\ldots1}_{n+1}
\]

Consider them all \( \mod n \)

\[\Rightarrow \text{there are only } n \text{ unique mod } n \text{ values} \quad (0, 1, 2, \ldots, n-1)\]

\[\Rightarrow \text{by PHP, some mod value occurs at least twice.} \]

\[a \mod b \]

\[a=q_1 \cdot n+r \quad b=q_2 \cdot n+r\]

\[a-b=(q_1-q_2) \cdot n \Rightarrow \text{a multiple of } n.\]
Generalized PHP: put \( n \) objects into \( k \) boxes, then at least one box has at least \( \lceil n/k \rceil \) objects.

\( \lceil x \rceil = "\text{ceiling}\" \) of \( x \)

= smallest integer at least as large as \( x \).

\( \lceil \frac{25}{10} \rceil = \lceil 2.5 \rceil = 3 \)

**Theorem:** Let \( x_1, x_2, \ldots, x_k \) be \( k \) integers where \( n = x_1 + x_2 + \cdots + x_k \)

and \( \bar{x} = \frac{n}{k} = \frac{x_1 + x_2 + \cdots + x_k}{k} \) is their average.

Then at least one \( x_i \) must be at least as large as \( \bar{x} \); i.e., \( x_i \geq \bar{x} \).

**Proof:** Assume for the sake of contradiction that no \( x_i \geq \bar{x} \).

Then all \( x_i < \bar{x} \).

\[
\begin{align*}
x_1 < \bar{x} \\
x_2 < \bar{x} \\
\vdots \\
x_k < \bar{x}
\end{align*}
\]

\[ \Rightarrow \quad x_1 + x_2 + \cdots + x_k < k \cdot \bar{x} \]

\[ \Rightarrow \quad \bar{x} > \frac{x_1 + x_2 + \cdots + x_k}{k} \]

\[ \Rightarrow \quad \bar{x} > \bar{x} \]

This is a contradiction, so the original assumption must be false.
Traveling Salesman Problem: 10 cities to visit, how many ways?

\[ 10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \]

"10 factorial"

10 choices second = 3,628,800
for first city

\[ 25! \approx 1.55 \times 10^{25} \]

⇒ exam : 37.6 million
possibilities every second
since beginning of time

\[ 10^{80} \text{ atoms in universe} \]
\[ 4.12 \times 10^{17} \text{ seconds since beginning of universe} \]

# ways to order n objects is n!
each ordering is referred to as a permutation
# permutations = n!
How many ways to visit 3 out of 10 cities?

\[
10 \cdot 9 \cdot 8 = \frac{10!}{7!} = \frac{10!}{(10-3)!}
\]

**Def:** Permutation: # permutation of n objects is

\[n! = n \cdot (n-1) \cdot (n-2) \ldots 3 \cdot 2 \cdot 1\]

**Def:** r-Permutation: # r-permutations of n objects is

\[nP_r = P(n, r) = \frac{n!}{(n-r)!} = n \cdot (n-1) \cdot \ldots \cdot (n-r+1)
\]

**Example** 10 packages to deliver

- How many ways? \(10! = 3,628,800\)
- 7 out of 10? \(\frac{10!}{(10-7)!} = \frac{10!}{3!} = \frac{10 \cdot 9 \cdot 8 \cdot 7!}{3!} = 604,800\)
- 6 on north side- together
  - 4 on south side- together
  - 2 \(\cdot 4! \cdot 6!\)
  - 4 south or north first
Combinations: an unordered subset of \( r \) out of \( n \) objects.

\[
\# \text{ combinations} = \binom{n}{r} = \binom{n,r} = \binom{2}{2} = \frac{n!}{r!(n-r)!}
\]

Proof: \( P(n,r) = \binom{n}{r} \cdot r! \)

\[
\binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}
\]