CS1800 Discrete Structures Practice Final

Instructions:

1. The exam is closed book and closed notes. You may not use a calculator or any other electronic device.

2. This is a two-hour exam worth 200 pts total. The points for each problem vary and are given in the problem statement.

3. Please look through the entire exam and verify that you have all pages numbered 1 through 14.

4. You should write your answers in the space provided; use the back sides of these sheets, if necessary.

5. Good luck!

<table>
<thead>
<tr>
<th>Problem</th>
<th>Points</th>
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<tbody>
<tr>
<td>Problem 1</td>
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<td>Problem 2</td>
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<td>Problem 3</td>
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<td>Problem 10</td>
<td>24</td>
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<td><strong>Total</strong></td>
<td><strong>200</strong></td>
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</table>

Name: ________________________ NU ID#: ________________________
Problem 1 [22 pts, (6.8.8)]: Decimal, Binary, Hex

(a) Fill in this table with the missing decimal, binary, or hexadecimal representations of the given numbers.

<table>
<thead>
<tr>
<th>decimal</th>
<th>binary</th>
<th>hexadecimal</th>
</tr>
</thead>
<tbody>
<tr>
<td>195</td>
<td>11000011</td>
<td>C3</td>
</tr>
<tr>
<td>170</td>
<td>10101010</td>
<td>AA</td>
</tr>
<tr>
<td>217</td>
<td>11011001</td>
<td>D9</td>
</tr>
</tbody>
</table>

(b) Fill in the table below with the 8-bit two’s complement representation of the given decimal integers.

<table>
<thead>
<tr>
<th>Decimal</th>
<th>Two’s Complement</th>
<th>Decimal</th>
<th>Two’s Complement</th>
</tr>
</thead>
<tbody>
<tr>
<td>35</td>
<td>00100011</td>
<td>−35</td>
<td>11011101</td>
</tr>
<tr>
<td>92</td>
<td>01011100</td>
<td>−92</td>
<td>10100100</td>
</tr>
</tbody>
</table>

(c) What is the smallest number of bits that can be used to represent all the numbers described in each of the following:

i. The integers from 0 to 1000 as unsigned (positive) binary numbers.

   Solution: 10

ii. The integers from 0 to 1,000,000 as unsigned (positive) binary numbers.

   Solution: 20

iii. The integers from −1000 to +1000 in twos-complement.

   Solution: 11

iv. The hexadecimal numbers from 0000 to FFFF as positive binary integers.

   Solution: 16
Problem 2 [21 pts, (4,10,4,3)]: Modular Arithmetic
Evaluate the following. Show your work.

i. 52 mod 7 =
   \textit{Solution: } 3

   \hspace{1cm} -52 \mod 7 = \hspace{1cm}
   \textit{Solution: } 4

ii. 180 \mod 13 = \hspace{1cm}
    \textit{Solution: } 11

   \hspace{1cm} 180^2 \mod 13 = \hspace{1cm}
   \textit{Solution: } 4

   \hspace{1cm} 180^4 \mod 13 = \hspace{1cm}
   \textit{Solution: } 3

   \hspace{1cm} 180^8 \mod 13 = \hspace{1cm}
   \textit{Solution: } 9

   \hspace{1cm} 180^{12} \mod 13 = \hspace{1cm}
   \textit{Solution: } 1
iii. Find the multiplicative inverse of 180 mod 13. That is, find an integer $x$ such that $0 \leq x < 13$ and $180 \cdot x \mod 13 = 1$.

Solution: 6

iv. If $n$ is any integer greater than 100, then $(n - 2)^6 \mod n =

Solution: $(-2)^6 \mod n = 64$
Problem 3 [16 pts, (4 each)]: Truth Tables, Boolean Formulae, and Circuits

Fill in the following table with the missing truth tables, Boolean formulae, and circuits.

<table>
<thead>
<tr>
<th>Circuit</th>
<th>Boolean Formula</th>
<th>Truth Table</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Circuit Diagram" /></td>
<td>((\neg A \land \neg B \land C) \lor (B \land C))</td>
<td></td>
</tr>
<tr>
<td><img src="image3" alt="Circuit Diagram" /></td>
<td>(A \land B \land C)</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>A B C</th>
<th>Out</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 0 0</td>
<td>0</td>
</tr>
<tr>
<td>0 0 1</td>
<td>0</td>
</tr>
<tr>
<td>0 1 0</td>
<td>0</td>
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<tr>
<td>0 1 1</td>
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<td>1 0 0</td>
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<td>1</td>
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</tbody>
</table>
Problem 4 [12 pts, (4 each)]: **Prime Factorization**

Give the prime factorization of each of the following integers. Write your answers in standard exponential form with prime factors in increasing order, e.g., $200 = 2^3 \cdot 5^2$.

i. $3250$

   **Solution:** $2^1 \cdot 5^3 \cdot 13^1$

ii. $4^4 \cdot 6^6$

   **Solution:** $2^{14} \cdot 3^6$

iii. $\frac{12!}{9! \cdot 3!}$

   **Solution:** $2^2 \cdot 5^1 \cdot 11^1$
Problem 5 [21 pts, (6,6,9)]: gcd, lcm, Euclidean Algorithm

(a) Find the gcd or lcm as indicated. These should be easy and done without the Euclidean algorithm.

i. gcd(4800, 44)
   Solution: 4

ii. lcm(512, 7)
    Solution: 3584

(b) Use the Euclidean algorithm to find gcd(288, 180). You must show your work.
   Solution: gcd(288, 180) = gcd(180, 108) = gcd(108, 72) = gcd(72, 36) = gcd(36, 0) = 36

(c) Let p and q be prime numbers with p < q. Evaluate the following.

i. gcd(p · q!, q · p!)
   Solution: q · p!

ii. gcd(p²q⁵, p³q⁴)
    Solution: p²q⁴

iii. lcm(p²q⁵, p³q⁴)
    Solution: p³q⁵
Problem 6 [24 pts, (12,12)]: Sets, Set Operations, Venn Diagrams.

(a) Recall that $\mathbb{N}$ is the set of natural numbers $0, 1, 2, \ldots$. In each of the following, show the set indicated by giving all of its elements, for example, $\{n \in \mathbb{N} \mid 1 < n < 5\} = \{2, 3, 4\}$.

i. $\{n \in \mathbb{N} \mid n^2 = k + 1 \text{ for some } k \in \mathbb{N} \text{ such that } 1 \leq k \leq 10\}$

Solution: $\{2, 3\}$

ii. $\mathcal{P}(S)$, the power set of $S$, where $S$ is the set $\{n \in \mathbb{N} \mid 1 \leq n \leq 10 \text{ and } n \mod 3 = 2\}$

Solution: $\{\emptyset, \{2\}, \{5\}, \{8\}, \{2, 5\}, \{2, 8\}, \{5, 8\}, \{2, 5, 8\}\}$

iii. $S \times S$ where $S$ is the set $\{n \in \mathbb{N} \mid 1 \leq n \leq 6 \text{ and } \gcd(n, 4) = 1\}$

Solution: $\{(1, 1), (1, 3), (1, 5), (3, 1), (3, 3), (3, 5), (5, 1), (5, 3), (5, 5)\}$

(b) Sets $A$, $B$, and $C$ are shown in the Venn diagrams below. For each part, color in the set indicated in the diagram to the right.

i. Solution: $A \cup \overline{B}$

ii. Solution: $(A \cup \overline{B}) \cap \overline{C}$

iii. Solution: $(\overline{A} \cup C) \cap (B \cup C)$
Problem 7 [30 pts, (3 each)]: Counting and Probability.

When my son visits for the holidays, we like to catch up on some of the TV series that me didn’t have time for during our academic semesters. This year, we plan to devote a few evenings to watching episodes of Bones and Castle. There are 5 seasons of Bones and 2 seasons of Castle available for instant download through Netflix. (There are a total of 106 episodes of Bones and 36 episodes of Castle.)

<table>
<thead>
<tr>
<th>Bones</th>
<th>Bones</th>
<th>Bones</th>
<th>Bones</th>
<th>Bones</th>
</tr>
</thead>
<tbody>
<tr>
<td>season 1</td>
<td>season 2</td>
<td>season 3</td>
<td>season 4</td>
<td>season 5</td>
</tr>
<tr>
<td>22 episodes</td>
<td>21 episodes</td>
<td>15 episodes</td>
<td>26 episodes</td>
<td>22 episodes</td>
</tr>
<tr>
<td>Castle</td>
<td>Castle</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>season 1</td>
<td>season 2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10 episodes</td>
<td>26 episodes</td>
<td></td>
<td></td>
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</tbody>
</table>

In parts (i) through (iv) below, give an answer in terms of permutations \( P(n, k) \) and/or combinations \( C(n, k) \) and then as a factorial or ratio of factorials or ratio of products and sums of integers.

i. On Monday, we plan to watch one episode from each season of Bones, first from season 1, then from season 2, and so on. How many ways can we do this?

Solution: \( C(22, 1) \cdot C(21, 1) \cdot C(15, 1) \cdot C(26, 1) \cdot C(22, 1) = 22 \cdot 21 \cdot 15 \cdot 26 \cdot 22 \)

ii. On Tuesday, we plan to watch 6 different episodes of Castle, one after another. In how many ways can we pick an ordered list of 3 different episodes from season 1 of Castle followed by an ordered list of 3 different episodes from season 2 of Castle?

Solution: \( P(10, 3) \cdot P(26, 3) = 10 \cdot 9 \cdot 8 \cdot 26 \cdot 25 \cdot 24 \)

iii. On Wednesday, my son will pick 7 different episodes (out of all 142) to play in order. I will be late so I’ll miss the first one. I also know that I really can’t sit around watching TV that long so I plan to leave and go to the gym while the last of the 7 episodes is playing. How many different lists of 7 episodes can my son choose such that I won’t miss any episodes of Castle?

Solution: \( 106 \cdot 105 \cdot P(140, 5) = 106 \cdot 105 \cdot 140 \cdot 139 \cdot 138 \cdot 137 \cdot 136 \)

iv. On Thursday, my son wants to choose 10 episodes of Castle to write about in his blog. He wants to include at least one episode from each season. How ways can he choose the 10 episodes?

Solution: We use inclusion-exclusion: \( C(36, 10) - C(26, 10) - C(10, 10) = \frac{36!}{(10! \cdot 26!)} - \frac{26!}{(16! \cdot 10!)} - 1 \)
v. Each episode of Bones runs for about an hour, and I would like to watch each of the 106 episodes during the next 6 weeks.
I am trying to make up a schedule giving the number of episodes I will watch each of the 6 weeks. Here is one example of a schedule.

<table>
<thead>
<tr>
<th>Week 1</th>
<th>Week 2</th>
<th>Week 3</th>
<th>Week 4</th>
<th>Week 5</th>
<th>Week 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>37</td>
<td>13</td>
<td>15</td>
<td>15</td>
<td>16</td>
</tr>
</tbody>
</table>

How many different schedules are there?

Solution: $C(106 + 5, 5) = 111!/(106! \cdot 5!)$

What is the largest number $N$ such that I know that I must watch at least $N$ episodes during one of those weeks?

Solution: $\lceil \frac{106}{6} \rceil = 18$

vi. After my son leaves on Saturday, I plan to randomly choose one episode to watch over lunch. There are three ways that I might do this. For each way, give the probability that the episode I choose is the 100th episode of Bones. (This is in season 5.)

I choose one episode at random from the 142 episodes.

Solution: $\frac{1}{142}$

I choose one of the 7 seasons (5 of Bones and 2 of Castle) at random and then choose an episode at random from that season.

Solution: $\frac{1}{7} \cdot \frac{1}{22}$

I toss a fair coin to decide between Bones and Castle then choose a season at random and then choose an episode at random.

Solution: $\frac{1}{2} \cdot \frac{1}{5} \cdot \frac{1}{22}$

vii. Now suppose I select two episodes at random (without replacement) from the set of all 142 episodes. What is the probability that one of the two episodes is from Castle, given the condition that at least one of the two selected episodes is from Bones.

Solution: Let $C$ be the event that one of the two episodes is from Castle. Let $B$ be the event at least one of the two episodes is from Bones.

$\text{Prob}(B) = (C(142, 2) - C(36, 2)) / C(142, 2)$

$\text{Prob}(C \text{ and } B) = C(36, 1) \cdot C(106, 1) / C(142, 2)$

$\text{Prob}(C \mid B) = C(36, 1) \cdot C(106, 1) / (C(142, 2) - C(36, 2)) = 36 \cdot 106 / (142 \cdot 141/2 - 36 \cdot 35/2)$
Problem 8 [12 pts, (6,6)]: Summations

i. Evaluate the following sum. To obtain full credit, your answer must be in the form of a single (correct) integer, and you must show your work.

$$\sum_{k=4}^{27} (3k + 2) =$$

**Solution:** By formula:

$$3\left(\sum_{k=1}^{27} k - \sum_{k=1}^{3} k\right) + 2(27 - 3) = 3 \cdot (27 \cdot 28/2 - 3 \cdot 4/2) - 48 = 1164$$

Or, since this sum is an arithmetic, you can just use Gauss’s trick.

$$S = (\text{first} + \text{last})(\text{numTerms})/2 = \frac{(3 \cdot 4 + 2 + 3 \cdot 27 + 2)(27 - 3)}{2} = 97 \cdot 12 = 1164$$

ii. Derive a formula in terms of $a$ and $n$ for the following sum. You must show your work.

$$\sum_{k=1}^{n} a^{2k+3} =$$

**Solution:** By formula:

$$a^3 \left(\sum_{k=1}^{n} (a^2)^k\right) = a^3 \frac{a^{2(n)} - 1}{a^2 - 1} = a^{2n+5} - a^5$$

Or, since this is a geometric sum, you can set

$$S = \sum_{k=1}^{n} a^{2k+3}$$

$$a^2S = \sum_{k=2}^{n+1} a^{2k+3}$$

$$(a^2 - 1)S = a^2 S - S = a^{2(n+1)+3} - a^{2+3} = a^{2n+5} - a^5$$

$$S = \frac{a^{2n+5} - a^5}{a^2 - 1}$$
Problem 9 [18 pts, (9,9)]: Mathematical Induction

i. Prove the following statement by mathematical induction, for all integers \( n \geq 1 \).

\[
\sum_{i=1}^{n} i(i + 1) = \frac{n(n + 1)(n + 2)}{3}.
\]

Solution: Let us denote by \( S(n) \) the above statement. Base case. We need to show \( S(1) \) is true, i.e. we need to show that \( 1 \cdot 2 = \frac{1 \cdot 2 \cdot 3}{3} \) which it trivially is. Inductive step. Let us assume the truth of \( S(n) \) for some \( n \geq 1 \) and now we need to show that \( S(n + 1) \) is true. From \( S(n) \) we get that

\[
\sum_{i=1}^{n} i(i + 1) = \frac{n(n + 1)(n + 2)}{3}.
\]

Extending the summation on the left-hand-side from \( n \) to \( n + 1 \) is equivalent to adding the term \( (n + 1)(n + 2) \) and so we get

\[
\sum_{i=1}^{n+1} i(i + 1) = (n+1)(n+2) + \frac{n(n + 1)(n + 2)}{3} = (n+1)(n+2)(1 + \frac{n}{3}) = \frac{(n + 1)(n + 2)(n + 3)}{3}.
\]

But this is exactly \( S(n + 1) \) and hence we are done.
ii. Using mathematical induction prove that $11^n - 6$ is divisible by 5 for all positive integers $n$.

**Solution:** Let us denote by $S(n)$ the statement: $11^n - 6$ is divisible by 5. Base case. We need to show $S(1)$ is true, i.e. we need to show that $11^1 - 6 = 5$ is divisible by 5 which it trivially is. Inductive step. Let us assume the truth of $S(n)$ for some $n \geq 1$ and now we need to show that $S(n + 1)$ is true. From $S(n)$ we get that there exists integer $m$ such that

$$11^n - 6 = 5m; \text{ or } 11^n = 5m + 6$$

Now, considering $S(n + 1)$ we see that

$$11^{n+1} - 6 = 11(5m + 6) - 6 = 55m + 60 = 5(11m + 12)$$

i.e. $11^{n+1} - 6$ is divisible by 5. But this is exactly $S(n + 1)$ and hence we are done.
Problem 10 [24 pts, (4 each)]: Graphs
Refer to the graph below for the following problems.

![Graph]

i. Show the adjacency list representation for this graph, where the vertices are ordered alphabetically and each adjacency list is also ordered alphabetically.

*Solution:*

\[
\begin{align*}
A &\rightarrow D, E \\
B &\rightarrow F, G \\
C &\rightarrow D, F \\
D &\rightarrow A, F, C \\
E &\rightarrow A, F \\
F &\rightarrow B, C, D, E, G \\
G &\rightarrow B, F
\end{align*}
\]

ii. Give the longest path that does not cross itself from vertex D to vertex G.

*Solution:*

\[D \rightarrow A \rightarrow E \rightarrow F \rightarrow B \rightarrow G\]

iii. List the vertices (separated by spaces) in the order they are visited in a Depth First Search that starts at vertex B. (Assume that DFS processes vertices alphabetically, when given the option of multiple vertices to explore.)

*Solution:*

\[BFCDAEG\]
iv. Give the path that is found from vertex B to vertex A using a *Depth First Search* that starts at vertex B. (Assume that DFS processes vertices alphabetically, when given the option of multiple vertices to explore.)

*Solution:*  

\[ BFCDA \]

v. List the vertices (separated by spaces) in the order they are visited in a *Breadth First Search* that starts at vertex B. (Assume that BFS processes vertices alphabetically, when given the option of multiple vertices to explore.)

*Solution:*  

\[ BFGCDEA \]

vi. Give the path that is found from vertex B to vertex A using a *Breadth First Search* that starts at vertex B. (Again assume that BFS processes vertices alphabetically, when given the option of multiple vertices to explore.)

*Solution:*  

\[ B \rightarrow F \rightarrow D \rightarrow A \]