DIGITAL IMAGE PROCESSING (COM-3371)
Week 4 - February 4, 2002

Topics:

• Processing images in frequency space
• 1-D Fourier transform
• 2-D Fourier transform
• 1-D Discrete Fourier transform
• 2-D Discrete Fourier transform
• Examples
• Properties of the 2D Fourier Transform
  • Separability
  • Translation
  • Periodicity and conjugate symmetry
• Rotation
• Addition
• Scaling
• Average value
• Convolution
• Fast Fourier Transform (FFT)
• Filtering in the frequency domain

Readings

• Chapter 4 (4.1-4.3, 4.4.1-4.4.3, 4.6) of text

Homework 2

Midterm
PROCESSING IMAGES IN FREQUENCY SPACE

AN IMAGE SCENE IS COMPOSED OF 2D SPATIAL FREQUENCY COMPONENTS \(\rightarrow\) AN IMAGE MAY BE BROKEN INTO THESE FREQUENCY COMPONENTS AND THEN RECONSTRUCTED FROM THEIR SUBSEQUENT SUMMATION

**Frequency transform** gives the ability to transform an image from the spatial to the frequency domain and back again

**ORIGINAL IMAGE** \(\rightarrow\) Frequency Transform \(\rightarrow\) **NEW IMAGE**

Frequency image displays the presence of frequency components in an original image by the brightness of points at respective locations:
- **horizontal frequency** is defined along the x-axis,
- **vertical frequency** is defined along the y-axis

The brightness of a point in the image corresponds to the amplitude of the frequency component.

**IMAGE FILTERING** may be carried out in the frequency domain

\(\rightarrow\) to remove a particular frequency band from an image, we may simply set the corresponding area of that frequency image to zero and transform the frequency image back to the spatial domain

\(\rightarrow\) by modifying image frequency components, periodic noise can be removed, edges can be enhanced, image can be blurred, etc.

\(\rightarrow\) most common frequency transforms: Fourier, Hadamard, Haar, Walsh
1-D FOURIER TRANSFORM

• BAPTISTE JOSEPH FOURIER (1768-1830)

1. In 1807 Fourier presented the results of his study of heat propagation and diffusion to the Institute of France
   ➔ Fourier claimed that any periodic signal could be represented by a series of sinusoids

(see Figure 4.1 in G&W)

Since then, Fourier work has been applied to many problems dealing with frequency analysis (mathematics, science, engineering)

• Let f(x) be a function of real variable x representing time or distance in one direction across an image.

\[ f(x) \text{ continuous} \]

(In a digitized image the values of x are not continuous, but discrete - based on pixel spacing; also, f(x) has quantized intensity values.)

• Fourier theorem states that it is possible to form any 1-D function f(x) as a summation of a series of sine and cosine terms of increasing frequency.

• The Fourier transform of the function f(x) is F(u):

\[ \mathcal{F}\{f(x)\} = F(u) = \int_{-\infty}^{+\infty} f(x)e^{-2\pi jux} \, dx \]  

(1)

frequency variable

\[ j = \sqrt{-1} \]

(in some books you will find i instead of j)

\[ e^{-j2\piux} = \cos(2\piux) - \sin(2\piux) \]

(Euler’s formula)
• Given $F(u)$, it is possible to recover the spatial domain function $f(x)$ by using the **Inverse Fourier Transform**:

$$\mathcal{F}^{-1}\{F(u)\} = f(x) = \int_{-\infty}^{+\infty} F(u)e^{2\pi jux} \, du$$

$$e^{j2\pi ux} = \cos(2\pi ux) + j\sin(2\pi ux)$$

• EQUATIONS 1 AND 2 ARE CALLED **THE FOURIER TRANSFORM PAIR**

$f(x)$ is generally a real function

$F(u)$ is generally complex:

$$F(u) = R(u) + jI(u)$$
It is often convenient to express Eq. 3 in the exponential form:

\[ F(u) = |F(u)|e^{j\phi(u)} \]

where

\[ |F(u)| = \sqrt{R^2(u) + I^2(u)} \]  \hspace{1cm} \text{(FOURIER SPECTRUM)} \hspace{1cm} (4)

and

\[ \phi(u) = \tan^{-1}\left(\frac{I(u)}{R(u)}\right) \]  \hspace{1cm} \text{(PHASE ANGLE)} \hspace{1cm} (5)

Another term:

\[ P(u) = |F(u)|^2 = R^2(u) + I^2(u) \]  \hspace{1cm} \text{(POWER SPECTRA)} \hspace{1cm} (6)
2-D FOURIER TRANSFORM

\( f(x, y) \leftrightarrow \text{continuous function} \)

\[
\mathcal{F}\{f(x, y)\} = F(u, v) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) e^{-2\pi j (ux + vy)} \, dx \, dy
\]

(7)

\[
\mathcal{F}^{-1}\{F(u, v)\} = f(x, y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F(u, v) e^{2\pi j (ux + vy)} \, du \, dv
\]

(8)

where \( u \) and \( v \) are frequency variables

**FOURIER SPECTRUM:**

\[
|F(u, v)| = \sqrt{R^2(u, v) + I^2(u, v)}
\]

(9)

**PHASE:**

\[
\phi(u, v) = \tan^{-1}\left[\frac{I(u, v)}{R(u, v)}\right]
\]

(10)

**POWER SPECTRUM:**

\[
P(u, v) = |F(u, v)|^2 = R^2(u, v) + I^2(u, v)
\]

(11)
• In the frequency domain, \( u \) represents the spatial frequency along the original image x axis, and \( v \) represents spatial frequency along the y axis.

• In the center of the image \( u \) and \( v \) have their origin

• Pixel that is further from the origin represents higher spatial frequency

**ANALOGY:**
Fourier transform can be compared to a glass prism. The prism separates light into various components, each depending on its wavelength (or frequency) content. Fourier transform can be viewed as a "mathematical prism" that separates a function into various components based on frequency content.
1-D DISCRETE FOURIER TRANSFORM

- Suppose that a continuous function \( f(x) \) is discretized into a sequence:

\[
\{ f(x_0), f(x_0 + \Delta x), f(x_0 + 2\Delta x), \ldots, f(x_0 + [N-1]\Delta x) \}
\]

by taking \( N \) samples \( \Delta x \) units apart

Let's define:

\[
f(x) = f(x_0 + \Delta x)
\]

where \( x \) now assumes the discrete values 0, 1, 2, 3, … \( N-1 \)

The sequence can be defined as:

\[
\{ f(0), f(1), f(2), \ldots, f(N-1) \}
\]

it defines any \( N \) uniformly spaced samples of continuous function
• The discrete 1-D Fourier transform is defined:

\[
F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x)e^{-j2\pi ux/N}
\]  

(12)

• The inverse 1-D discrete Fourier Transform:

\[
f(x) = \sum_{u=0}^{N-1} F(u)e^{j2\pi ux/N}
\]

(13)

• The Fourier spectrum, phase angle and power spectrum are given by Equations 4-6.

**Example:**

*Example:* To illustrate Eqs. (3.2-2) and (3.2-3), consider the function shown in Fig. 3.5(a). Sampling at the argument values \(x_0 = 0.5, x_1 = 0.75, x_2 = 1.0,\) and \(x_3 = 1.25\) — and redefining the argument as discussed above — produces the discrete function shown in Fig. 3.5(b).

Application of Eq. (3.2-2) to the resulting four samples yields the following sequence of steps:

\[
F(0) = \frac{1}{4} \sum_{x=0}^{3} f(x)\exp[0]
\]

\[
= \frac{1}{4} [f(0) + f(1) + f(2) + f(3)]
\]

\[
= \frac{1}{4} (2 + 3 + 4 + 4)
\]

\[
= 3.25
\]
and

\[ F(1) = \frac{1}{4} \sum_{x=0}^{3} f(x) \exp\left[-j2\pi x/4\right] \]

\[ = \frac{1}{4} (2e^{0} + 3e^{-j\pi/2} + 4e^{-j\pi} + 4e^{-j3\pi/2}) \]

\[ = \frac{1}{4} (-2 + j) . \]

where the last step follows from Euler's formula. Continuing with this procedure gives

\[ F(2) = -\frac{1}{4} [1 + j0] \]

and

\[ F(3) = -\frac{1}{4} [2 + j] . \]

All values of \( f(x) \) contribute to each of the four terms of the discrete Fourier transform. Conversely, all terms of the transform contribute in forming the inverse transform via Eq. (3.2-3). The procedure for obtaining the inverse is analogous to the one just described for computing \( F(u) \).

The Fourier spectrum is obtained from the magnitude of each of the transform terms; that is,

\[ |F(0)| = 3.25 \]

\[ |F(1)| = \left[ \left( \frac{2}{4} \right)^{2} + \left( \frac{1}{4} \right)^{2} \right]^{1/2} = \frac{\sqrt{3}}{4} \]

\[ |F(2)| = \left[ \left( \frac{1}{4} \right)^{2} + \left( \frac{0}{4} \right)^{2} \right]^{1/2} = \frac{1}{4} \]

and

\[ |F(3)| = \left[ \left( \frac{2}{4} \right)^{2} + \left( \frac{1}{4} \right)^{2} \right]^{1/2} = \frac{\sqrt{3}}{4} . \]
2-D DISCRETE FOURIER TRANSFORM

- A continuous function $f(x,y)$ is sampled with divisions of width $\Delta x$ and $\Delta y$ in the x and y axis, respectively.

- As in 1-D case, the discrete function $f(x,y)$ represents samples of the function $f(x_0+m\Delta x, y_0+n\Delta y)$ for $m=0,1,2,\ldots,M-1$ and $n=0,1,2,\ldots,N-1$.

- When images are sampled in a square array, $M=N$.

- The 2-D discrete Fourier Transform is defined:

$$F(u, v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)}$$ \hspace{1cm} (14)

- The inverse 2-D discrete Fourier Transform:

$$f(x, y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)}$$ \hspace{1cm} (15)

- The Fourier spectrum, phase angle and power spectrum are given by Equations 9-11.

EXAMPLES:

1-D FT \(\Rightarrow\) Fig. 4.2
2-D FT \(\Rightarrow\) Fig. 4.3
More Examples

vertical pattern; 8 cycles/256 pixels

original image  FFT Power Spectrum

vertical pattern; 32 cycles/256 pixels

original image  FFT Power Spectrum
horizontal pattern; 32 cycles/256 pixels

original image

FFT Power Spectrum

horizontal pattern+ vertical Pattern; 32 cycles/256 pixels

original image

FFT Power Spectrum